Analytic conformal extensions of asymptotically flat space-times

1983 J. Phys. A: Math. Gen. 16 2187

(http://iopscience.iop.org/0305-4470/16/10/015)

J. Phys. A: Math. Gen. 16 (1983) 2187–2190. Printed in Great Britain

Analytic conformal extensions of asymptotically flat space–times

Bernd Schmidt and Martin Walker

Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Strasse 1, D-8046 Garching, Federal Republic of Germany

Received 22 November 1982

Abstract. The conformal structure of Schwarzschild's space–time is analytically extended through the hypersurfaces at null infinity. The space–times on the other side of infinity are conformally isometric to Schwarzschild's space–time with negative mass. Analogous analytic conformal extensions of any analytic asymptotically flat space–times can be made.

1. The analytic extension through $\mathscr{I}^+$

In retarded null coordinates, Schwarzschild's metric assumes the form

$$ds^2 = (1-2m/r)\,du^2 + 2\,du\,dr - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{1}$$

where $m>0$, $-\infty<u<\infty$, $0<r<\infty$, $(\theta,\phi)\in\mathbb{S}^2$. The conformal structure determined by the metric (1) consists of all metrics $\Omega^2\,ds^2$ with $\Omega\neq 0$. Defining $l=1/r$ and multiplying the metric (1) by l^2 gives

$$l^2\,ds^2 = l^2(1-2ml)\,du^2 - 2\,du\,dl - (d\theta^2+\sin^2\theta\,d\phi^2). \tag{2}$$

The hypersurface $\mathscr{I}^+$ at future null infinity is given by $l=0$ in (2). The metric (2) can be analytically extended by allowing l to take on negative values, and in this way we have analytically extended the conformal structure of the original Schwarzschild metric.

What is the space–time on the other side of $\mathscr{I}^+$? Undoing the conformal rescaling above, we now have the metric

$$ds'^2 = (1-2ml)\,du^2 - 2l^{-2}\,du\,dl - l^{-2}(d\theta^2+\sin^2\theta\,d\phi^2) \tag{3}$$

where $-\infty<l<0$. Introducing new coordinates $r'=-l^{-1}$, $v'=u$ and defining $m'=-m$, the metric (3) becomes

$$ds'^2 = (1-2m'/r')\,dv'^2 - 2\,dv'\,dr' - r'^2(d\theta^2+\sin^2\theta\,d\phi^2) \tag{4}$$

where $0<r'<\infty$ and $-\infty<v'<\infty$. We recognise (4) as again being Schwarzschild's metric, but in advanced null coordinates and, since $m'<0$, with negative mass. We see also that the hypersurface $\mathscr{I}^+$ at future null infinity of the original Schwarzschild space–time now appears as the hypersurface $\mathscr{I}^-$ at past null infinity for the Schwarzschild space–time with negative mass. The space–times described by the metrics (1) and (4) are disconnected. On the level of their conformal structure, however, each is an extension of the other.

2. Global considerations

It is entertaining to construct possible maximal conformal extensions of Schwarzschild's space–time. We begin with the well known Kruskal–Penrose diagram (Penrose 1968), figure 1, in which the shaded domain is that covered by the retarded null coordinates of metric (1). The corresponding diagram for the metric (4) is that of figure 2. Figure 3 shows the extension described in § 1.

There are exactly two maximally unambiguous conformal extensions of the space–time of figure 3. These are shown in figure 4. Each of the space–times in figure 4 can be maximally extended in infinitely many different ways. A particularly simple conformally inextensible conformal extension of Schwarzschild's space–time is that

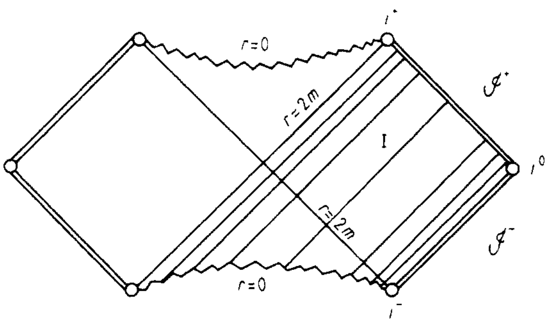

Figure 1. The Kruskal–Penrose diagram of maximally extended Schwarzschild space–time. The double lines indicate the hypersurfaces at past and future null infinity. The open circles indicate that these points are absent from the manifolds. The topology of the space–time is that of the figure ($\mathbb{R}^2$) times the the two-sphere ($\mathbb{S}^2$).

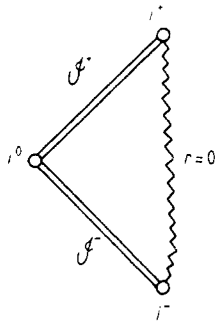

Figure 2. The Kruskal–Penrose diagram of negative-mass Schwarzschild space–time.

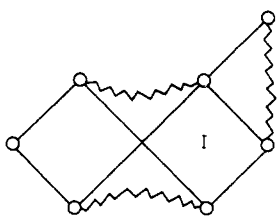

Figure 3. Region I of figure 1 has been conformally extended through $\mathscr{I}^+$.

of figure 5. The space–time of figure 5 is invariant under discrete translations in the time direction which preserve the pattern. Identifications can be made using this translation. In particular, the smallest inextensible conformal extension of Schwarzschild's space–time is shown in figure 6. The largest inextensible conformal extension of Schwarzschild's space–time (whose diagram we shall not attempt to draw) is the universal covering space of the space–time of figure 6.

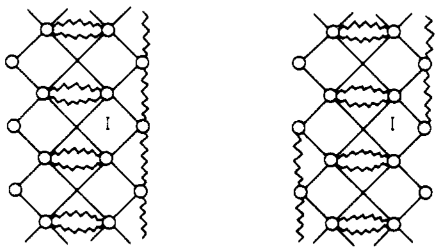

Figure 4. Figure 3 can be conformally analytically extended in two different ways before identification ambiguities can arise.

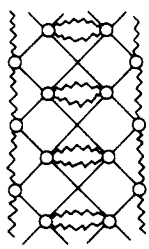

Figure 5. A simple maximal analytic extension. The pattern extends infinitely far into past and future.

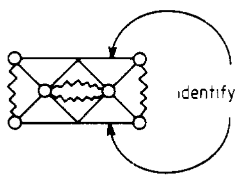

Figure 6. The simplest (acausal) analytic conformal extension of Schwarzschild space–time.

3. Miscellaneous remarks

The conformal structure of the analytic conformal extensions of figure 1 is not continuous at the points labelled i^-, i^0, i^+. This is in contrast to the case of flat space–time, in which the universal covering space of the maximal conformal extension is the Einstein universe (Kuiper 1949, Schmidt 1974). The regular point i^0 at space-like infinity of one conformal copy of flat space–time in the Einstein universe is at the same time the point $i^{-/+}$ at past/future time-like infinity for that conformal copy of flat space–time reached by extending through $\mathscr{I}^{+/-}$ of the original copy.

We see an analogous situation in the case of conformal extensions of Schwarzschild's space–time. The point i^0 at space-like infinity of the original Schwarzschild space–time is at the same time the point at past/future time-like infinity of the negative mass conformal extension through $\mathscr{I}^{+/-}$ of the original space–time.

For space–times admitting convergent Bondi expansions, analogous conformal extensions can be made. The space–times on the other side of $\mathscr{I}$ are conformally isometric to the space–times obtained from the originals by reversal of the signs of coefficients of odd powers of r^{-1} in the Bondi expansions. Examples are the space–times of Reissner and Nordstrøm, Kerr and Bonnor and Swaminaryan (see Bičák 1968).

Acknowledgment

We thank Jürgen Ehlers for comments on the manuscript.

References

Bičák J 1968 *Proc. R. Soc.* A **302** 201
Kuiper N H 1949 *Ann. Math.* **50** 916
Penrose R 1968 *Battelle Rencontres* ed C M De Witt and J A Wheeler (New York: Benjamin) p 208
Schmidt B G 1974 *Commun. Math. Phys.* **36** 73